The metric in the superspace of Riemannian metrics and its relation to gravity*

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Abstract

The space of all Riemannian metrics is infinite-dimensional. Nevertheless a great deal of usual Riemannian geometry can be carried over. The superspace of all Riemannian metrics shall be endowed with a class of Riemannian metrics; their curvature and invariance properties are discussed. Just one of this class has the property to bring the lagrangian of General Relativity into the form of a classical particle's motion. The signature of the superspace metric depends in a non-trivial manner on the signature of the original metric, we derive the corresponding formula. Our approach is a local one: the essence is a metric in the space of all symmetric rank-two tensors, and then the space becomes a warped product of the real line with an Einstein space.

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1 THE SUPERSPACE

Let $n \geq 2$, n be the dimension of the basic Riemannian spaces. Let M be an n-dimensional differentiable manifold with an atlas x of coordinates x^i, $i = 1, \ldots, n$. The signature s (= number of negative eigenvalues) shall be fixed; let V be the space of all Riemannian metrics $g_{ij}(x)$ in M with signature s, related to the coordinates x^i. This implies that isometrical metrics in M are different points in V in general. The V is called superspace, its points are the Riemannian metrics. The tangent space in V is the vector space

$$T = \{h_{ij}(x) | x \in M,\ h_{ij} = h_{ji}\} \tag{1}$$

the space of all symmetric tensor fields of rank 2. All considerations are local ones, so we may have in mind one single fixed coordinate system in M.

2 COORDINATES IN SUPERSPACE

Coordinates should possess one contravariant index, so we need a transformation of the type

$$y^A = \mu^{Aij} g_{ij}(x) \tag{2}$$

such that the y^A are the coordinates for V. To have a defined one–to–one correspondence between the index pairs $(i,\ j)$ and the index A we require

$$A = 1, \ldots N = n(n+1)/2\,,$$

and $A = 1, \ldots N$ corresponds to the pairs

$$\begin{aligned}(1,1),\ (2,2), \ldots (n,n),\ (1,2),\ (2,3), \ldots (n-1,n),\ (1,3),& \\ \ldots (n-2,n), \ldots (1,n)&\end{aligned} \tag{3}$$

consecutively. $(i,\ j)$ and $(j,\ i)$ correspond to the same A. We make the ansatz

$$\begin{aligned}\mu^{Aij} = \mu_{Aij} = b \text{ for } i \neq j, \quad c \text{ for } i = j& \\ 0 \text{ if } (i,j) \text{ does not correspond to } A&\end{aligned} \tag{4}$$

and require the usual inversion relations

$$\mu^{Aij}\mu_{Bij} = \delta^A_B \quad \text{and} \quad \mu^{Aij}\mu_{Akl} = \delta^{(i}_k \delta^{j)}_l \,. \tag{5}$$

Bracketed indices are to be symmetrized, which is necessary because of symmetry of the metric g_{ij}. Inserting ansatz (4) into (5) gives $c^2 = 1$, $b^2 = 1/2$. Changing the sign of b or c only changes the sign of the coordinates, so we may put

$$c = 1, \qquad b = 1/\sqrt{2}\,. \tag{6}$$

The object μ^{Aij} is analogous to the Pauli spin matrices relating two spinorial indices to one vector index.

3 METRIC IN SUPERSPACE

The metric in the superspace shall be denoted by H_{AB}, it holds

$$H_{AB} = H_{BA} \tag{7}$$

and the transformed metric is

$$G^{ijkl} = H_{AB}\mu^{Aij}\mu^{Bkl}\,, \quad H_{AB} = \mu_{Aij}\mu_{Bkl}G^{ijkl}\,. \tag{8}$$

From (4) and (7) it follows that

$$G^{ijkl} = G^{jikl} = G^{klij}\,. \tag{9}$$

The inverse to H_{AB} is H^{AB}, and we define

$$G_{ijkl} = H^{AB}\mu_{Aij}\mu_{Bkl} \tag{10}$$

which has the same symmetries as (9). We require G^{ijkl} to be a tensor and use only the metric $g_{ij}(x)$ to define it. Then the ansatz

$$G^{ijkl} = z\, g^{i(k}\, g^{l)j} + \alpha\, g^{ij}\, g^{kl} \tag{11}$$

$$G_{ijkl} = v\, g_{i(k}\, g_{l)j} + \beta\, g_{ij}\, g_{kl} \tag{12}$$

where v, z, α and β are constants, is the most general one to fulfil the symmetries (9). One should mention that also curvature-dependent constants could have heen introduced. The requirement that H_{AB} is the inverse to H^{AB} leads via (8,10) to

$$G_{ijkl} G^{klmp} = \delta_i^{(m} \delta_j^{p)} \,. \tag{13}$$

The requirement that G^{ijkl} is a tensor can be justified as follows: Let a curve $y^A(t)$, $0 < t < 1$ in V be given, then its length is

$$\sigma = \int_0^1 \left(H_{AB} \frac{dy^A}{dt} \frac{dy^B}{dt} \right)^{1/2} dt$$

i.e., with (2) and (8)

$$\sigma = \int_0^1 \left(G^{ijkl} \frac{dg_{ij}}{dt} \frac{dg_{kl}}{dt} \right)^{1/2} dt \,. \tag{14}$$

A coordinate transformation in $M : x^i \to \epsilon x^i$ changes $g_{ij} \to \epsilon^{-2} g_{ij}$. We now require that σ shall not be changed by such a transformation. Then α and z are constant real numbers. Inserting (11,12) into (13) gives $v\,z = 1$, hence $z \neq 0$. By a constant rescaling we get

$$v = z = 1 \tag{15}$$

and then (11,12,13) yield

$$\alpha \neq \frac{-1}{n} \,, \quad \beta = \frac{-\alpha}{1 + \alpha n} \,. \tag{16}$$

So we have got a one-parameter set of metrics in V. Eq. (16) fulfils the following duality relation: with $f(\alpha) = -\alpha/(1 + \alpha n)$, $f(f(\alpha)) = \alpha$ holds for all $\alpha \neq -1/n$.

$$\begin{aligned} G^{ijkl} &= g^{i(k}\, g^{l)j} + \alpha\, g^{ij}\, g^{kl} \\ G_{ijkl} &= g_{i(k}\, g_{l)j} + \beta\, g_{ij}\, g_{kl} \,. \end{aligned} \tag{17}$$

It holds: For $\alpha = -1/n$, the metric H_{AB} is not invertible.
Indirect proof: G^{ijkl} depends continuously on α, so it must be the case with the inverse. But

$$\lim_{\alpha \to -1/n}$$

applied to G_{ijkl} gives no finite result. Contradiction.

4 Signature of the superspace metric

Let S be the signature of the superspace metric H_{AB}. S depends on α and s. For convenience we define

$$\Theta = 0, \quad (\alpha > -1/n) \qquad 1, \quad (\alpha < -1/n) \,. \tag{18}$$

From continuity reasons it follows that S is a function of Θ and s: $S = S(\Theta, s)$. If we transform $g_{ij} \to -g_{ij}$ i.e., $s \to n-s$, then H_{AB} is not changed, i.e.,

$$S(\Theta, s) = S(\Theta, n-s) \,. \tag{19}$$

We transform g_{ij} to diagonal form as follows

$$g_{11} = g_{22} = \ldots = g_{ss} = -1, \quad g_{ij} = \delta_{ij} \quad \text{otherwise} \,. \tag{20}$$

4.1 Signature for $\Theta = 0$

To calculate $S(0, s)$ we may put $\alpha = 0$ and get with (8,11,15)

$$H_{AB} = \mu_{Aij} \, \mu_{Bkl} \, g^{ik} \, g^{jl} \tag{21}$$

which is a diagonal matrix. It holds $H_{11} = \ldots = H_{nn} = 1$ and the other diagonal components are ± 1. A full estimate gives in agreement with (19)

$$S(0, s) = s(n-s) \,. \tag{22}$$

4.2 Signature for $\Theta = 1$

To calculate $S(1, s)$ we may put $\alpha = -1$ and get

$$H_{AB} = \mu_{Aij} \, \mu_{Bkl} \left(g^{ik} \, g^{jl} - g^{ij} \, g^{kl} \right) \,. \tag{23}$$

For $A \le n < B$, $H_{AB} = 0$, i.e., the matrix H_{AB} is composed of two blocks. For A, $B \le n$ we get

$$H_{AB} = 0 \quad \text{for} \quad A = B, \qquad 1 \quad \text{for} \quad A \neq B$$

a matrix which has the $(n-1)$-fold eigenvalue 1 and the single eigenvalue $1-n$. For $A, B > n$ we have the same result as for the case $\alpha = 0$, i.e., we get $S(1, s) = 1 + s(n-s)$.

4.3 Result

The signature of the superspace metric is

$$S = \Theta + s(n-s)\,. \tag{24}$$

5 SUPERCURVATURE

We use exactly the same formulae as for finite-dimensional Riemannian geometry to define Christoffel affinities Γ^A_{BC} and Riemann tensor R^A_{BCD}. Using (4) we write all equations with indices $i, j = 1, \dots n$. Then each pair of covariant indices i, j corresponds to one contravariant index A. The following formulae appear:

$$\frac{\partial g^{ij}}{\partial g_{km}} = -g^{i(k}\, g^{m)j} \tag{25}$$

$$\Gamma^{ijklmp} = -\frac{1}{2} g^{i(k} g^{l)(m} g^{p)j} - \alpha g^{ij} g^{k(m} g^{p)l} - \frac{1}{2} g^{j(k} g^{l)(m} g^{p)i} \tag{26}$$

and, surprisingly independent of α we get

$$\Gamma^{klmp}_{ij} = -\delta^{(k}_{(i}\, g^{l)(m}\, \delta^{p)}_{j)}\,. \tag{27}$$

Consequently, also Riemann- and Ricci tensor do not depend on α:

$$R^{klmpij}_{rs} = \frac{1}{2}\left(\delta^{(k}_{(r}\, g^{l)(m} g^{p)(i}\, \delta^{j)}_{s)} - \delta^{(k}_{(r}\, g^{l)(i} g^{j)(m}\, \delta^{p)}_{s)}\right)\,. \tag{28}$$

Summing over $r = m$ and $s = p$ we get

$$R^{klij} = \frac{1}{4}\left(g^{ij} g^{kl} - n g^{k(i} g^{j)l}\right)\,. \tag{29}$$

The Ricci tensor has one eigenvalue 0. Proof: It is not invertible because it is proportional to the metric for the degenerated case $\alpha = -1/n$, cf. sct. 3.

The co–contravariant Ricci tensor reads

$$R^{ij}_{kl} = G_{klmp} R^{mpij} = \frac{1}{4}\left(g^{ij} g_{kl} - n\delta^{(i}_{k}\delta^{j)}_{l}\right)\,, \tag{30}$$

and the curvature scalar is

$$R = -\frac{1}{8}n(n-1)(n+2)\,. \tag{31}$$

The eigenvector to the eigenvalue 0 of the Ricci tensor is g_{ij}. All other eigenvalues equal $-n/4$, and the corresponding eigenvectors can be parametrized by the symmetric traceless metrices, i.e. the multiplicity of the eigenvalue $-n/4$ is $(n-1)(n+2)/2$.

6 SUPERDETERMINANT

We define the superdeterminant

$$H = \det H_{AB}\,. \tag{32}$$

H is a function of g, α and n which becomes zero for $\alpha = -1/n$, cf. sct. 3. We use eqs. (8) and (17) to look in more details for the explicit value of H. The formal calculation for $n = 1$ leads to

$$H = H_{11} = G^{1111} = g^{11}g^{11} + \alpha g^{11}g^{11} = (1+\alpha)g^{-2}\,.$$

Multiplication of g_{ij} with ϵ gives $g \to \epsilon^n g$, $H_{AB} \to \epsilon^{-2}H_{AB}$ and $H \to \epsilon^{-n(n+1)}H$. So we get in an intermediate step

$$H = H_1\, g^{-n-1} \tag{33}$$

where H_1 is the value of H for $g = 1$. H_1 depends on α and n only. To calculate H_1 we put $g_{ij} = \delta_{ij}$ and get via $H_{ij} = \delta_{ij} + \alpha$, $H_{Ai} = 0$ for $A > n$, and $H_{AB} = \delta_{AB}$ for $A, B > n$ finally

$$H_1 = 1 + \alpha n\,. \tag{34}$$

This is in agreement with the $n = 1$-calculation.

7 GRAVITY

Now, we come to the main application: The action for gravity shall be expressed by the metric of superspace. We start from the metric

$$ds^2 = dt^2 - g_{ij}\, dx^i\, dx^j \tag{35}$$

$i, j = 1, \ldots n$ with positive definite g_{ij} and $x^0 = t$. We define the second fundamental form K_{ij} by

$$K_{ij} = \frac{1}{2}\, g_{ij,0}\,. \tag{36}$$

The Einstein action for (35) is

$$I = -\int {}^*R\, \frac{1}{2}\, \sqrt{g}\, d^{n+1}x \tag{37}$$

where $g = \det g_{ij}$ and ${}^*R$ is the $(n+1)$-dimensional curvature scalar for (35). Indices at K_{ij} will be shifted with g_{ij}, and $K = K^i_i$. With (36) we get

$$(K\sqrt{g})_{,0} = (K_{,0} + K^2)\sqrt{g}\,. \tag{38}$$

This divergence can be added to the integrand of (37) without changing the field equations. It serves to cancel the term $K_{,0}$ of I. So we get

$$I = \int \frac{1}{2} \left(K^{ij} K_{ij} - K^2 + R \right) \sqrt{g}\, d^{n+1}x \tag{39}$$

where R is the n-dimensional curvature scalar for g_{ij}. We make now the ansatz for the kinetic energy

$$W = \frac{1}{2} G^{ijmp} K_{ij} K_{mp} = \frac{1}{2} \left(K^{ij} K_{ij} + \alpha K^2 \right)\,. \tag{40}$$

Comparing (40) with (39) we see that for $\alpha = -1$ (surprisingly, this value does not depend on n)

$$I = \int \left(W + \frac{R}{2} \right) \sqrt{g}\, d^{n+1}x \tag{41}$$

holds. Because of $n \geq 2$ this value α gives a regular superspace metric. (For $n = 1$, eq. (37) is a divergence, and $\alpha = -1$ gives not an invertible superspace-metric.)

Using the μ^{Aij} and the notations $z^A = \mu^{Aij}\, g_{ij}\,/\,2$ and $v^A = dz^A/dt$ we get from (40,41)

$$I = \int \frac{1}{2}\left(H_{AB}v^A v^B + R(z^A)\right)\sqrt{g}\, d^{n+1}x \tag{42}$$

i.e., the action has the classical form of kinetic plus potential energy. The signature of the metric H_{AB} is $S = 1$. This can be seen from eqs. (18,24).

8 CONCLUSION

In eq. (42), Einstein gravity is given in a form to allow canonical quantization: The momentum v^A is replaced by $-i\partial/\partial z^A$ $\;(\hbar = 1)$, and then the Wheeler - DeWitt equation for the world function $\psi(z^A)$ appears as Hamiltonian constraint in form of a wave equation:

$$\left(\Box - R(z^A)\right)\psi = 0\,. \tag{43}$$

After early attempts in [1], the Wheeler - DeWitt equation has often been discussed, especially for cosmology, see e.g. [2-5]. Besides curvature, matter fields can be inserted as potential, too. It is remarkable that exactly for Lorentz and for Euclidean signatures in (35) (positive and negative definite g_{ij} resp.) the usual D'Alembert operator ($S = 1$) in (43) appears. For other signatures in (35), (43) has at least two timelike axes.

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